A note on: Optimal ordering policy for stock-dependent demand under progressive payment scheme

Christoph H. Glock
Department of Law and Economics, Technische Universität Darmstadt
(glock@pscm.tu-darmstadt.de)
Jörg M. Ries
Department of Law and Economics, Technische Universität Darmstadt
(ries@pscm.tu-darmstadt.de)
Kurt Schwindl
University of Applied Sciences Würzburg-Schweinfurt
(kurt.schwindl@fhws.de)

Abstract: In a recent paper, Soni and Shah [Soni, H., Shah, N. H., 2008. Optimal ordering policy for stock-dependent demand under progressive payment scheme. European Journal of Operational Research 184, 91-100] developed a model to find the optimal ordering policy for a retailer with stock-dependent demand and a supplier that offers a progressive payment scheme to the retailer. This note corrects some errors in the formulation of the model of Soni and Shah. It also extends their work by assuming that the credit interest rate of the retailer may exceed the interest rate charged by the supplier. Numerical examples illustrate the benefits of these modifications.

Keywords: Economic order quantity (EOQ); stock-dependent demand; progressive credit periods; trade credit

Introduction

Recently, Soni and Shah (2008) developed a model to find the optimal ordering policy for a retailer with stock-dependent demand and a supplier that offers a progressive payment scheme to the retailer. The authors assumed that if the retailer settles its balance before time M, the supplier charges no interest to the retailer, whereas in case when the retailer settles its balance between times M and N with $M < N$, the supplier charges an interest rate Ic_1 on the outstanding balance. In the case when the retailer pays after time N, the supplier charges an interest rate Ic_2 with $Ic_2 > Ic_1$. Revenues the retailer receives from sales may be deposited in an interest-bearing account until the account is settled completely[1], where an interest is earned at the rate of Ie. Soni and Shah (2008) assumed that in case the retailer is not able to settle its unpaid balance at time M (or N), s/he will settle as much of the unpaid balance as possible at these points in time. The work of Soni and Shah (2008) was extended by Teng et al. (2011) who included additional aspects in the model, such as deterioration, limited capacity and non-zero ending inventory, and by Shah et al. (2011) who considered a variable retailer selling price in addition to the extensions made by Teng et al. (2011).

In developing their model, Soni and Shah (2008) implicitly assumed that the interest rate charged by the supplier in the first credit period, Ic_1, always exceeds the credit interest rate of the retailer, Ie. We note that this is not necessarily the case in practice. Instead, the interest rates charged by the supplier, Ic_1 and Ic_2, and the credit interest rate of the retailer, Ie, usually depend on the investment opportunities of the respective companies. Ie could thus represent the interest rate the

[1] Thus, we do not consider investment decisions which are not related to the lot sizing problem.

retailer could realize by depositing money in an interest-bearing account, but it could also represent the profit that the retailer can gain from other business activities or its opportunity cost of capital (Summers and Wilson, 2002). The same applies to the interest rates charged by the supplier. It is clear that the ratios of Ie to Ic_1 and Ic_2 thus depend on the individual business environments of the supplier and the retailer, and that Ie could possibly exceed Ic_1 and Ic_2. If Ie exceeds Ic_1, for example, it may not be reasonable for the retailer to settle its unpaid balance at time M, as assumed in Soni and Shah (2008). Instead, it would be better to keep the sales revenue in an interest-bearing account or to invest it elsewhere, and to settle the unpaid balance when the interest charged by the supplier exceeds the returns from interest. This note extends the work of Soni and Shah (2008) by explicitly assuming that the case $Ie > Ic_1$ may occur in addition to the other cases studied by the authors. However, the case $Ie > Ic_2$, where the retailer never pays the supplier, is excluded.

Depending on the ratio of the interest rates Ic_1 and Ie and the time when the retailer sells off the entire production lot, ten different cases arise, which are summarized in Table 1. The cases that were not treated by Soni and Shah (2008) will be discussed briefly in the following, and further some errors contained in their work will be corrected. We adopt the assumptions and notations used in Soni and Shah (2008) hereafter, unless it is stated otherwise.

Ratio of T, M and N	Ratio of interest rates	Unpaid balance	Account settled	Treated in subcase
$\boldsymbol{T \leq M}$	$Ic_1 \geq Ie$	---	M	1.1
$\boldsymbol{T \leq M}$	$Ic_1 < Ie$	---	N	1.2
$\boldsymbol{M < T \leq N}$	$Ic_1 \geq Ie$	$U_1 = 0$	M	2.1
$\boldsymbol{M < T \leq N}$	$Ic_1 \geq Ie$	$U_1 > 0$	$M + z$	2.2
$\boldsymbol{M < T \leq N}$	$Ic_1 < Ie$	---	N	2.3
$\boldsymbol{T > N}$	$Ic_1 \geq Ie$	$U_1 = 0$	M	3.1
$\boldsymbol{T > N}$	$Ic_1 \geq Ie$	$U_2 = 0$	$M + z$	3.2
$\boldsymbol{T > N}$	$Ic_1 \geq Ie$	$U_2 > 0$	$N + z$	3.3
$\boldsymbol{T > N}$	$Ic_1 < Ie$	$U_3 = 0$	N	3.4
$\boldsymbol{T > N}$	$Ic_1 < Ie$	$U_3 > 0$	$N + z$	3.5

Table 1: Cases for settling the unpaid balance

Modified model

Subcase 1.1: This case is discussed as ‘Case 1’ in Soni and Shah (2008).

Subcase 1.2: For $T \leq M$ and $Ie > Ic_1$, the retailer achieves a financial benefit from postponing the refund and investing the sales revenue until time N. Between times M and N, s/he has to pay interest to the supplier. However, due to $Ie > Ic_1$, the interest earned exceeds the interest paid within the specified time period. The interest earned per year can be calculated as:

$$IE_{1,2} = \frac{PIe}{T}\left(\int_0^T R(t)tdt + Q(N-T)\right) = \frac{PIea}{b^2T}\left(e^{bT}(b(N-T)+1) - bN - 1\right) \tag{1}$$

The overall interest charged between M and N amounts to:

$$IC_{1,2} = \frac{Ic_1}{T} CQ(N - M) = \frac{CIc_1 a}{bT} (e^{bT} - 1)(N - M) \tag{2}$$

The total costs are calculated from Eqs. (1) and (2) by considering ordering cost and inventory holding cost in addition:

$$TC_{1,2} = \frac{A}{T} + \frac{ha}{b^2 T} (e^{bT} - bT - 1) + \frac{CIc_1 a}{bT} (e^{bT} - 1)(N - M) - \frac{PIea}{b^2 T} (e^{bT}(b(N - T) + 1) - bN - 1) \tag{3}$$

The optimal solution to Eq. (3) is the solution of the following non-linear equation:

$$\frac{dTC_{1,2}}{dT} = -\frac{A}{T^2} - \frac{ah(e^{bT} - bT - 1)}{b^2 T^2} + \frac{ah(e^{bT} - 1)}{bT} + \frac{CIc_1 a(bTe^{bT} - e^{bT} + 1)(N - M)}{bT^2} + \frac{IePa(e^{bT} - bT - 1)}{bT} + \frac{IePa(e^{bT}(b(N - T) + 1) - bN - 1)}{b^2 T^2} = 0 \tag{4}$$

which minimizes $TC_{1,2}$ provided that the second derivation with respect to T is

$$\frac{d^2 TC_{1,2}}{dT^2} = \frac{2A}{T^3} - \frac{2CIc_1 ae^{bT}(N - M)}{T^2} + \frac{2CIc_1 a(e^{bT} - 1)(N - M)}{bT^3} + \frac{CIc_1 abe^{bT}(N - M)}{T} - \frac{2a(e^{bT} - 1)}{bT^2} + \frac{hae^{bT}}{T} + \frac{2ha(e^{bT} - bT - 1)}{b^2 T^3} - \frac{PIea(1 - bT)}{T} - \frac{2PIea(e^{bT}(b(N - T) + 1) - bN - 1)}{b^2 T^3} - \frac{2PIea(e^{bT} - bT - 1)}{T^2 b} > 0, \text{ for all } T. \tag{5}$$

Subcase 2.1: In the case when $M < T \leq N$ and $Ie \leq Ic_1$, the retailer settles as much of the unpaid balance as possible at time M to minimize interest payments. The first subcase assumes that the sum of sales revenue and interest earned by time M is sufficient to settle the unpaid balance, i.e. $U_1 = 0$. The interest earned until time M is formulated as follows (note that this formulation corrects an error in Soni and Shah's Eq. (3.11)):

$$IE_{2,1} = \frac{PIe}{T} \int_0^M R(t) t \mathrm{d}t = \frac{PIea}{b^2 T} e^{b(T - M)} (e^{bM} - bM - 1) \tag{6}$$

As the retailer does not have to pay interest to the supplier in this subcase (i.e., $IC_{2,1} = 0$), the total costs amount to:

$$TC_{2,1} = \frac{A}{T} + \frac{ha}{b^2 T} (e^{bT} - bT - 1) - \frac{PIea}{b^2 T} e^{b(T - M)} (e^{bM} - bM - 1) \tag{7}$$

The optimal solution to Eq. (7) is the solution of the following non-linear equation:

$$\frac{dTC_{2,1}}{dT} = -\frac{A}{T^2} - \frac{ha(e^{bT} - bT - 1)}{b^2 T^2} + \frac{ha(e^{bT} - 1)}{bT} + \frac{PIeae^{b(T - M)}(e^{bM} - bM - 1)}{b^2 T^2} = 0 \tag{8}$$

which minimizes $TC_{2,1}$ provided that the second derivation with respect to T is

$$\frac{d^2 TC_{2,1}}{dT^2} = \frac{2A}{T^3} - \frac{2ha(e^{bT} - 1)}{bT^2} + \frac{hae^{bT}}{T} + \frac{2ha(e^{bT} - bT - 1)}{b^2 T^3} - \frac{2PIeae^{b(T - M)}(e^{bM} - bM - 1)}{b^2 T^3} > 0, \text{ for all } T. \tag{9}$$

Subcase 2.2: In this subcase, the sum of sales revenue and interest earned by time M is not sufficient to settle the balance completely, i.e. $U_1 > 0$. Thus, the retailer has to pay interest on U_1. Interest earned is the same as the one given in Eq. (6). In calculating the unpaid balance U_1, Soni and Shah (2008) assumed that $U_1 = CQ - (PR(M)M + IE_2)$, where $R(t)$ denotes the stock-dependent demand rate. Since the demand rate decreases in t due to a decreasing inventory level, we note that $PR(M)M$ underestimates the sales revenue of the retailer, since $R(M) < R(M-\Delta)$ for $\Delta > 0$. As a consequence, U_1 has to be reformulated as follows:

$$U_1 = CQ - \left(P\int_0^M R(t)\mathrm{d}t + PIe\int_0^M R(t)t\mathrm{d}t\right) \tag{10}$$

Furthermore, the authors mentioned that the "retailer will have to pay interest on un-paid balance […] at the rate of Ic_1 at time M to the supplier". However, we note that after the account has been partially settled at time M, the retailer has no money left to pay interest in advance. We therefore modify Soni and Shah's approach and assume that when $U_1 > 0$ and $Ie \leq Ic_1$, the retailer transfers each dollar s/he earns after time M directly to the supplier to minimize interest payments (note that this is a reasonable assumption if the supplier and the buyer use common electronic payment technology with low transaction cost. See Goyal et al., 2007 for a similar assumption). For the case when the unpaid balance cannot be settled at time M, but before time N, it follows that the interest paid as given in Eq. (3.17) of the Soni and Shah-paper can be reformulated as follows:

$$IC_{2,2} = \frac{Ic_1}{T}\int_M^{M+z}\big(U_1 - PR(t)(t-M)\big)\mathrm{d}t \tag{11}$$

where z is the time period that is needed to settle the remaining account U_1, derived by comparing the unpaid balance with the outstanding earnings between M and M+z, which leads to $z = -\frac{1}{b}\log\left(1 - U_1/\left(P\frac{a}{b}e^{b(T-M)}\right)\right)$. Accordingly, M+z denotes the point in time when the unpaid balance has been completely settled, with $z > 0$ and $M+z < T$. The total costs for this case again can be derived as the sum of ordering, inventory holding and interest costs less interest earned. Due to the complexity of the precise expression of z and the consequent interest cost $IC_{2,2}$, the explicit optimality conditions for Subcase 2.2 were omitted in this case.[2] The value of T can be approximated numerically with arbitrary precision (e.g. with the help of the bisection method).

Subcase 2.3: This subcase occurs when $M < T \leq N$ and $Ie > Ic_1$, and is identical to Subcase 1.2.

Subcase 3.1: This subcase occurs when $T > N$, $Ie \leq Ic_1$ and $U_1 = 0$, and is identical to Subcase 2.1.

Subcase 3.2: This subcase occurs when $T > N$, $Ie \leq Ic_1$, $U_1 > 0$ and $U_2 = 0$, and is identical to Subcase 2.2.

Subcase 3.3: In this subcase, with $T > N$ and $Ie \leq Ic_1$, the retailer is not able to pay off the total purchase cost at times M or N. Thus, s/he will settle as much of the balance as is possible at times M and N. Between times M and N, the sales revenue is invested, and the supplier charges interest

[2] The precise expressions can be obtained by the authors upon request.

on the outstanding balance U_1 with interest rate Ic_1. Afterwards, as in Subcase 2.2, the retailer transfers each dollar s/he earns directly to the supplier who charges interest on the gradually reducing unpaid balance U_2 at the interest rate Ic_2. As the retailer partially settles the account in M and N, s/he is able to realize interest earnings in the period [0, N], which can be calculated as:

$$IE_{3,3} = \frac{PIe}{T}\left(\int_0^M R(t)t\mathrm{d}t + \int_{\mathrm{M}}^N R(t)(t-M)\mathrm{d}t\right) \tag{12}$$

The unsettled balance U_2 (at time N) calculated by Soni and Shah (2008) again underestimates the sales revenue of the retailer. Further, while estimating the interest earnings between times M and N, the authors neglected the time period the revenue is kept in the account. Therefore, U_2 has to be reformulated as follows:

$$U_2 = U_1\big(1 + Ic_1(N-M)\big) - \left(P\int_M^N R(t)\,\mathrm{d}t + PIe\int_M^N R(t)(t-M)\,dt\right) \tag{13}$$

where U_1 is the unpaid balance at time M as given by Eq. (10). Consequently, the interest payable per year, $IC_{3,3}$, is given as:

$$IC_{3,3} = \frac{Ic_1}{T}U_1(N-M) + \frac{Ic_2}{T}\int_N^{N+z}\big(U_2 - PR(t)(t-N)\big)\,\mathrm{d}t \tag{14}$$

where z is the time period that is needed to settle the remaining account U_2, derived by comparing the unpaid balance with the outstanding earnings between N and N+z, which leads to $z = -\frac{1}{b}\log\left(1 - U_2/\left(P\frac{a}{b}e^{b(T-N)}\right)\right)$. Accordingly, N+z denotes the point in time when the unpaid balance is settled, with $z > 0$ and $N+z \le T$. The objective function for Subcase 3.3 has the same structure and solution procedure as the one given in Subcase 2.2, with the exceptions that $IC_{2,2}$ needs to be substituted by $IC_{3,3}$ and that the interests earnings $IE_{3,3}$ have to be considered. Again, a near-optimal solution can be calculated numerically.

Subcase 3.4: If the interest rate of the retailer, Ie, exceeds the interest charges of the supplier for the first credit period, Ic_1, s/he will again not settle the account before time N. Instead, the retailer invests the revenues from sales for the period M to N. As the unpaid balance U_3 is zero in this subcase, the account is completely settled at time N. Thus, the interest earned is given as:

$$IE_{3,4} = \frac{PIe}{T}\int_0^N R(t)tdt = \frac{PIea}{b^2T}e^{b(T-N)}(e^{bN} - bN - 1) \tag{15}$$

The interest charges in the period [M, N] are the same as those given in Eq. (2). Thus, the total costs for this subcase are formulated as:

$$TC_{3,4} = \frac{A}{T} + \frac{ha}{b^2T}(e^{bT} - bT - 1) + \frac{CIc_1a}{bT}(e^{bT} - 1)(N-M) - \frac{\mathrm{PIe}a}{b^2T}e^{b(T-N)}(e^{bN} - bN - 1) \tag{16}$$

The optimal solution to Eq. (17) is the solution of the following non-linear equation:

$$\frac{dTC_{3,4}}{dT} = -\frac{A}{T^2} + \frac{ha\left(e^{bT}-bT-1\right)}{b^2T^2} - \frac{CIc_1a\left(e^{bT}-1\right)(N-M)}{bT^2} + \frac{PIeae^{b(T-N)}\left(e^{bN}-bN-1\right)}{b^2T^2} = 0 \quad (17)$$

which minimizes $TC_{2,1}$ provided that the second derivation with respect to T is

$$\frac{d^2TC_{3,4}}{dT^2} = \frac{2A}{T^3} - \frac{ha}{T} - \frac{2ha\left(e^{bT}-bT-1\right)}{b^2T^3} + \frac{2C\mathrm{Ic}_1a\left(e^{bT}-1\right)(N-M)}{bT^3} - \frac{2PIeae^{b(T-N)}\left(e^{bN}-bN-1\right)}{b^2T^3} > 0, \text{ for all } T. \quad (18)$$

Subcase 3.5: For the case when $Ie > Ic_1$ and $U_3 > 0$, the account is partially settled at time N, and afterwards the unpaid balance is continuously reduced by transferring each dollar earned from sales to the supplier until the balance has been settled completely. The interest earnings until time N are the same as those given in Eq. (15). In addition, the unpaid balance at time N equals:

$$U_3 = CQ\left(1 + Ic_1(N-M)\right) - \left(P\int_0^N R(t)\,\mathrm{d}t + IeP\int_0^N R(t)t\,\mathrm{d}t\right) \quad (19)$$

The interest charges amount to:

$$IC_{3,5} = \frac{Ic_1}{T}CQ(N-M) + \frac{Ic_2}{T}\int_N^{N+z}\left(U_3 - PR(t)(t-N)\right)dt \quad (20)$$

where z is the time period that is needed to settle the remaining account U_3, derived by comparing the unpaid balance with the outstanding earnings between N and N+z, which leads to $z = -\left(\frac{1}{b}\right)\log(1 - U_3/\left(P\frac{a}{b}e^{b(T-N)}\right)$. Accordingly, N+z denotes the point in time when the unpaid balance is settled, with $z > 0$ and $N+z \leq T$. The objective function for Subcase 3.5 has the same structure and solution procedure as the one given in Subcase 2.2, with the exception that $IC_{2,2}$ needs to be substituted by $IC_{3,5}$ and the interest earnings $IE_{3,4}$ have to be considered. Again, a near-optimal solution can be calculated numerically.

Numerical examples

To illustrate the behavior of our model, we consider the parametric values shown in Table 1 and the payment policies of the retailer introduced above. The numerical examples (cf. Tables 2 and 3) indicate that:

1. For a fixed value of the scale parameter b, an increase in the first credit period leads to higher order quantities and demand rates. The total cost, nevertheless, is reduced as M adopts higher values. An increase in the second credit period, likewise, results in higher order quantities, higher demand and lower total costs, the cost reduction, however, is less than in the former case.
2. An inverse interest structure with $Ie > Ic_1$ does not affect the lot size policy. However, it affects the optimal payment policy of the retailer, who may choose a different point in time to settle the balance. In contrast to the model of Soni and Shah (2008) (cf. TC_1 in Table 2), the presented payment policy (cf. TC_2 in Table 2) may reduce the total costs of the buyer.
3. An increase in the sensitivity of demand in the on-hand inventory level leads to a higher inventory level, higher demand and lower total costs.

Table 1: Model parameters

a	=	1000	minimum demand
A	=	200	ordering cost per order
b	=	4.00	scale parameter of the demand function
C	=	20	unit purchase cost
h	=	0.20	inventory holding cost per unit and year
Ic_1	=	0.10	interest rate per year for the first credit period
Ic_2	=	0.18	interest rate per year for the second credit period
Ie	=	0.14	interest rate on deposits for the retailer
M	=	15	first permissible credit period
N	=	30	second permissible credit period
P	=	40	unit selling price

Table 2: Effect of M and b on the decision variables with N = 30/365

	M		
b	15/365	20/365	25/365
3.5			
Q	965.29	1039.38	1131.38
R	4378.53	4637.83	4959.82
TC_1	672.96	606.93	534.91
TC_2	659.13	594.81	527.41
4.0			
Q	1116.61	1225.58	1360.78
R	5466.45	5902.30	6443.13
TC_1	662.99	587.02	502.59
TC_2	648.02	573.82	494.35
4.5			
Q	1261.50	1435.61	1899.20
R	6676.74	7460.25	9546.40
TC_1	651.11	563.10	460.61
TC_2	635.30	548.96	451.02

Table 3: Effect of N and b on the decision variables with M = 15/365

	N		
b	30/365	35/365	40/365
3.5			
Q	965.29	1026.73	1078.05
R	4378.53	4593.57	4773.18
TC_1	672.96	648.05	627.06
TC_2	659.13	631.84	609.62
4.0			
Q	1116.61	1177.85	1241.84
R	5466.45	5711.41	5967.34
TC_1	662.99	637.50	616.75
TC_2	648.02	620.62	599.47
4.5			
Q	1261.50	1375.41	1456.18
R	6676.74	7189.36	7552.80
TC_1	651.11	625.49	626.54
TC_2	635.30	608.30	589.21

Conclusion

In this note, we corrected several errors in the work of Soni and Shah (2008) and modified some of its assumptions to increase the model's applicability. In contrast to the original paper, our note assumed that the deposit interest rate of the buyer, Ie, may exceed the interest charged by the supplier in the first credit period, Ic_1. Such a scenario may occur if the buyer and the supplier have access to different sources of funding or different investment opportunities, for example. This may result in different interest rates that are used at both actors. Numerical examples illustrated the behavior of our model and showed that the optimal payment policy, which depends on the current interest structure, may lead to lower cost without changing the lot-size policy itself. From a managerial perspective, carefully considering the prevailing interest structure, which governs the payment policy without affecting ordering decisions, is indispensable for minimizing total cost. The results of this paper also illustrate the close linkage between operational and financial aspects in supply chain management, which should be considered by employing integrated planning approaches.

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